Time-dependent constrained Hamiltonian systems and Dirac brackets

1996 J. Phys. A: Math. Gen. 29 6843

(http://iopscience.iop.org/0305-4470/29/21/016)

J. Phys. A: Math. Gen. **29** (1996) 6843–6859. Printed in the UK

Time-dependent constrained Hamiltonian systems and Dirac brackets

Manuel de León†‖, Juan C Marrero‡¶ and David Martín de Diego§+

† Instituto de Matemáticas y Física Fundamental, Consejo Superior de Investigaciones Científicas, Serrano 123, 28006 Madrid, Spain

‡ Departamento de Matemática Fundamental, Facultad de Matemáticas, Universidad de la Laguna, La Laguna, Tenerife, Canary Islands, Spain

§ Departamento de Economía Aplicada Cuantitativa, Facultad de Ciencias Económicas y Empresariales, UNED, 28040 Madrid, Spain

Received 4 March 1996, in final form 12 June 1996

Abstract. In this paper the canonical Dirac formalism for time-dependent constrained Hamiltonian systems is globalized. A time-dependent Dirac bracket which reduces to the usual one for time-independent systems is introduced.

1. Introduction

The aim of this paper is to globalize the Dirac approach to constrained Hamiltonian systems [6, 18, 19, 2, 16, 4, 20] by using modern tools of differential geometry. If we start with a singular time-independent Lagrangian function $L(q^A, \dot{q}^A)$ defined on a phase space of velocities TQ, the Hamiltonian energy h_1 is well-defined on the submanifold $M_1 = Leg(TQ)$, provided that L is almost regular. In this case, the canonical formalism of Dirac [6] proceeds as follows. M_1 is defined by the vanishing of some functions (called primary constraints) ϕ^a on T^*Q, and these primary constraints have to be preserved in time yielding new (secondary) constraints. Eventually, a final constraint submanifold is obtained. The Dirac constraint algorithm was globalized by Gotay and Nester [8–10] (see [4] for a recent review). By using the second-class constraints, one constructs the so-called Dirac bracket $\{\ ,\ \}_{\mathrm{D}}$, which is a modification of the canonical Poisson bracket $\{\ ,\ \}$ on the phase space T^*Q. The reason is that the evolution of an observable f is simply written as $\dot{f} \approx \{f, H\}_{\mathrm{D}}$, for an appropiate prolongation H of h_1, and, moreover $\{\ ,\ \}_{\mathrm{D}}$ transforms second-class constraints into Casimir functions. These results are crucial for quantization. This approach was globalized in a recent paper [14] by using almost product structures.

One may think that the extension of the theory to the time-dependent case is straightforward, a matter of pure technicalities. However, we realized that the geometry is more involved. The first point is that one has to use a cosymplectic geometry instead of a symplectic one, because the evolution of an observable is measured by using a modified Hamiltonian vector field (called the evolution vector field for obvious reasons). The second point is that, when we are in the presence of second-class constraints, the corresponding

‖ E-mail address: mdeleon@pinar1.csic.es
¶ E-mail address: jcmarrero@ull.es
+ E-mail address: dmartin@sr.uned.es

0305-4470/96/216843+17$19.50

Dirac bracket is more involved, and it becomes the usual one if the Lagrangian is time-independent.

The paper is structured as follows. Sections 2 and 3 are devoted to introducing the geometrical machinery: cosymplectic and adapted almost product structures and their associated Dirac brackets. In section 4 we review the constraint algorithm developed in [5] and [11]. In section 5 we apply the geometrical results to the analysis of a constrained time-dependent Hamiltonian system. For clarity, we first consider Lagrangians admitting a global dynamics (where there are no secondary constraints). We obtain the corresponding Dirac bracket which gives the evolution of an observable (with respect to an appropiate prolongation of the Hamiltonian function) and transforms second-class constraints into Casimir functions. Next, we analyse constrained systems with secondary constraints in section 6 and an example is studied in order to check the effectiveness of our method. In section 7 the time-independent case is recovered and, in section 8, the Dirac bracket for affine Lagrangian systems is obtained. The paper is completed by a technical appendix.

2. Cosymplectic vector spaces and manifolds

Let V be a real vector space of dimension $2m+1$, η a 1-form on V and ω a 2-form on V. Then the triple (V, η, ω) is called a cosymplectic vector space if $\eta \wedge \omega^m \neq 0$.

If V is a real vector space, η is a 1-form and ω is a 2-form on V, we define the linear map

$$\chi_{\eta,\omega} : V \longrightarrow V^* \qquad v \longrightarrow \chi_{\eta,\omega}(v) = i_v\omega + (\eta(v))\eta$$

where V^* is the dual space of V. We have the following proposition.

Proposition 2.1 ([1]). $\chi_{\eta,\omega}$ is a linear isomorphism iff either (V, η, ω) is a cosymplectic vector space in the case where V is odd dimensional, or (V, ω) is a symplectic vector space in the case where V is even dimensional.

Thus, if (V, η, ω) is a cosymplectic vector space then there exists a unique $\mathcal{R} \in V$ such that $\eta(\mathcal{R}) = 1$ and $i_{\mathcal{R}}\omega = 0$. In fact, $\mathcal{R} = \chi_{\eta,\omega}^{-1}(\eta)$. $\mathcal{R}$ is called the Reeb vector of the cosymplectic vector space (V, η, ω).

Let W be a subspace of a cosymplectic vector space (V, η, ω). We define the orthocomplement of W in V with respect to (η, ω) as the subspace $W^\perp$ given by

$$W^\perp = \{v \in V / (i_v\omega - \eta(v)\eta)_{|W} = 0\}\,. \tag{2.1}$$

We obtain the following proposition.

Proposition 2.2. If W is a subspace of a cosymplectic vector space (V, η, ω) then $\dim V = \dim W + \dim W^\perp$ and $(W^\perp)^\perp = \{v - 2\eta(v)\mathcal{R} \ / \ v \in W\}$. Moreover, if $W \cap W^\perp = \{0\}$, we have $V = W \oplus W^\perp$.

Now, suppose that M is a smooth $(2m+1)$-dimensional manifold. M is said to be almost cosymplectic if a 1-form η and a 2-form ω on M exist such that for all $x \in M$ the triple (T_xM, η_x, ω_x) is a cosymplectic vector space, where T_xM is the tangent space to M at x. If the 1-form η and the 2-form ω are closed, M is called cosymplectic.

Let (M, η, ω) be an almost cosymplectic manifold. Denote by $\mathcal{R}$ the Reeb vector field of (M, η, ω) and by $\chi_{\eta,\omega} : TM \longrightarrow T^*M$ the corresponding smooth vector bundle isomorphism.

By means of $\chi_{\eta,\omega}$ one can associate with every function $f \in C^\infty(M)$ the Hamiltonian vector field X_f which is defined by (see [1, 3]):

$$X_f = \chi_{\eta,\omega}^{-1}(\mathrm{d}f - \mathcal{R}(f)\eta) \iff i_{X_f}\eta = 0 \quad i_{X_f}\omega = \mathrm{d}f - \mathcal{R}(f)\eta\,. \tag{2.2}$$

Furthermore, the gradient vector field grad f and the evolution vector field E_f are given by

$$\operatorname{grad} f = \mathcal{R}(f)\mathcal{R} + X_f \qquad E_f = \mathcal{R} + X_f \,. \tag{2.3}$$

If (M, η, ω) is a cosymplectic manifold, there exists on $C^\infty(M)$ a Poisson bracket defined by $\{f, g\} = \omega(X_f, X_g)$. The symplectic leaves of this Poisson structure are precisely the leaves of the integrable distribution $\ker \eta$ (see [1]). It should be noted that

$$E_f(g) = \mathcal{R}(g) + \{g, f\}$$

gives the evolution of g with respect to the Hamiltonian function f. So, in order to get the evolution of an observable we need a Poisson bracket and a vector field.

3. Almost product structures adapted to precosymplectic structures

An almost product structure on a manifold M is a tensor field F of type $(1, 1)$ on M such that $F^2 = \mathrm{id}$. The manifold M will be called an almost product manifold [15]. If we set $\mathcal{A} = \frac{1}{2}(\mathrm{id} + F)$, $\mathcal{B} = \frac{1}{2}(\mathrm{id} - F)$, then $\mathcal{A}$ and $\mathcal{B}$ are complementary projectors, i.e. $\mathcal{A} + \mathcal{B} = \mathrm{id}$, $\mathcal{A}^2 = \mathcal{A}$, $\mathcal{B}^2 = \mathcal{B}$, $\mathcal{AB} = \mathcal{BA} = 0$. We denote by $Im\mathcal{A}$ and $Im\mathcal{B}$ the corresponding complementary distributions. Hence $TM = Im\mathcal{A} \oplus Im\mathcal{B}$. We denote by $\mathcal{A}^*$ and $\mathcal{B}^*$ the transpose operators, and $Im\mathcal{A}^*$ and $Im\mathcal{B}^*$ will be their corresponding images.

Next, we will introduce the notion of almost product structure adapted to an almost precosymplectic structure. First, let us recall the definition of almost precosymplectic structure (see [5]).

Definition 3.1. Let M be a manifold, η a 1-form and ω a 2-form on M. The pair (η, ω) is said to be an almost precosymplectic structure on M if $\eta \wedge \omega^r \neq 0$ and $\omega^{r+1} = 0$. Moreover, if η and ω are closed then the pair (η, ω) is called a precosymplectic structure.

Remark 3.2. If (η, ω) is an almost precosymplectic structure on M then there exists at least a vector field $\mathcal{R}$ on M such that $\eta(\mathcal{R}) = 1$ and $i_{\mathcal{R}}\omega = 0$ (see [5]).

Definition 3.3. Let (η, ω) be an almost precosymplectic structure on a manifold M and $\mathcal{R}$ a vector field such that $\eta(\mathcal{R}) = 1$ and $i_{\mathcal{R}}\omega = 0$. An almost product structure $(\mathcal{A}, \mathcal{B})$ on M is said to be adapted to the triple $(\eta, \omega, \mathcal{R})$ if

$$\ker \omega \cap \ker \eta = \ker \mathcal{A} \qquad \mathcal{R} = \mathcal{A}(\mathcal{R}) \,.$$

Remark 3.4. An almost product structure $(\mathcal{A}, \mathcal{B})$ is adapted to the triple $(\eta, \omega, \mathcal{R})$ if and only if

$$\ker \omega = \ker(\mathcal{A} - \eta \otimes \mathcal{R}) \qquad \mathcal{A}^*\eta = \eta \,.$$

Moreover, if $\bar{F}$ is the $(1, 1)$-tensor field on M given by $\bar{F} = \mathcal{A} - \mathcal{B} - \eta \otimes \mathcal{R}$, a direct computation proves that $\bar{F}^2 = \mathrm{id} - \eta \otimes \mathcal{R}$. Thus, the triple $(\bar{F}, \eta, \mathcal{R})$ is an almost paracontact structure on M (see [17]).

Let $(\mathcal{A}, \mathcal{B})$ be an almost product structure on M adapted to a triple $(\eta, \omega, \mathcal{R})$. Consider the linear mapping $\chi_{\eta,\omega} : \mathfrak{X}(M) \longrightarrow \Lambda^1(M)$ defined by

$$\chi_{\eta,\omega}(X) = i_X\omega + (\eta(X))\eta \,.$$

Thus, $\chi_{\eta,\omega}(\mathcal{R}) = \eta$, and $\chi_{\eta,\omega}$ induces an isomorphism of $C^\infty(M)$-modules $\chi_{\eta,\omega} : Im\mathcal{A} \longrightarrow Im\mathcal{A}^*$.

Using $\chi_{\eta,\omega}$ one can associate with every function $f \in C^\infty(M)$ an $\mathcal{A}$-Hamiltonian vector field $X_{(f,\mathcal{A})}$ which is given by

$$X_{(f,\mathcal{A})} = \chi_{\eta,\omega}^{-1}(\mathcal{A}^*(\mathrm{d}f) - \mathcal{R}(f)\eta) \in Im\mathcal{A} \tag{3.1}$$

or, equivalently,

$$X_{(f,\mathcal{A})} \in Im\mathcal{A} \qquad i_{X_{(f,\mathcal{A})}}\omega = \mathcal{A}^*(\mathrm{d}f) - \mathcal{R}(f)\eta \qquad i_{X_{(f,\mathcal{A})}}\eta = 0\,. \tag{3.2}$$

Also, the $\mathcal{A}$-gradient vector field $(\mathrm{grad}\, f)_\mathcal{A}$ and the $\mathcal{A}$-evolution vector field $E_{(f,\mathcal{A})}$ are given by

$$(\mathrm{grad}\, f)_\mathcal{A} = \mathcal{R}(f)\mathcal{R} + X_{(f,\mathcal{A})} \qquad E_{(f,\mathcal{A})} = \mathcal{R} + X_{(f,\mathcal{A})}\,. \tag{3.3}$$

Now, we define a bracket of functions as follows:

$$\{f, g\}_\mathcal{A} = \omega(X_{(f,\mathcal{A})}, X_{(g,\mathcal{A})}) \tag{3.4}$$

where $f, g \in C^\infty(M)$; $\{\ ,\ \}_\mathcal{A}$ satisfies all the properties of a Poisson bracket except the Jacobi identity. Moreover, if $\bar{\mathcal{A}}$ and $\bar{\mathcal{B}}$ are given by $\bar{\mathcal{A}} = \mathcal{A} - \eta \otimes \mathcal{R}$, $\bar{\mathcal{B}} = \mathcal{B} + \eta \otimes \mathcal{R}$, it is clear that the pair $(\bar{\mathcal{A}}, \bar{\mathcal{B}})$ is an almost product structure adapted to the almost presymplectic 2-form ω and the bracket of functions on M defined by the almost product structure $(\bar{\mathcal{A}}, \bar{\mathcal{B}})$ is just $\{\ ,\ \}_\mathcal{A}$. Thus, using the results for presymplectic structures in [7, 14], we have the following proposition.

Proposition 3.5. Let (η, ω) be an almost precosymplectic structure on M and $\mathcal{R}$ a vector field such that $i_\mathcal{R}\omega = 0$ and $i_\mathcal{R}\eta = 1$. Suppose that the 2-form ω is closed and that $(\mathcal{A}, \mathcal{B})$ is an almost product structure adapted to the triple $(\eta, \omega, \mathcal{R})$. Then the bracket $\{\ ,\ \}_\mathcal{A}$ defined by the almost product structure satisfies the Jacobi identity if and only if the almost product structure $(\bar{\mathcal{A}}, \bar{\mathcal{B}})$ is integrable, where $\bar{\mathcal{A}} = \mathcal{A} - \eta \otimes \mathcal{R}$ and $\bar{\mathcal{B}} = \mathcal{B} + \eta \otimes \mathcal{R}$.

In this case, if H is a Hamiltonian function on M we have $E_{(H,\mathcal{A})}(f) = \mathcal{R}(f) + \{f, H\}_\mathcal{A}$, for an observable f. Thus, $\dot{f} = \mathcal{R}(f) + \{f, H\}_\mathcal{A}$ is the evolution of f provided that $E_{(H,\mathcal{A})}$ gives the dynamics. The point is to construct a suitable adapted almost product structure.

4. The constraint algorithm

Let Q be a m-dimensional manifold and denote by $\tau_Q : TQ \longrightarrow Q$ the canonical projection. If (q^A), $1 \leqslant A \leqslant m$, are local coordinates on a neighbourhood U of Q, we denote by $(q^A, \dot{q}^A)$, $1 \leqslant A \leqslant m$, the induced coordinates on TU. Q will be the configuration manifold for a time-dependent Lagrangian system with Lagrangian function $L : \mathbb{R} \times TQ \longrightarrow \mathbb{R}$. If the Hessian matrix

$$\left(\frac{\partial^2 L}{\partial \dot{q}^A \partial \dot{q}^B} \right)$$

is regular, L is called regular, and singular or degenerate otherwise. The energy function associated with L is defined by $E_L = CL - L$, where C is the Liouville vector field on TQ. The Poincaré–Cartan 1-form and 2-form are respectively defined by $\alpha_L = J^*(\mathrm{d}L) - E_L\mathrm{d}t$ and $\Omega_L = -\mathrm{d}\alpha_L$, where t is the standard coordinate on $\mathbb{R}$. Here, J is the natural extension to $\mathbb{R} \times TQ$ of the canonical almost tangent structure of TQ. We define a 2-form $\omega_L = -\mathrm{d}(J^*(\mathrm{d}L))$ such that $\Omega_L = \omega_L + \mathrm{d}E_L \wedge \mathrm{d}t$.

If L is regular, the pair $(\mathrm{d}t, \omega_L)$ is a cosymplectic structure whose Reeb vector field is denoted by $\mathcal{R}_L$. Thus, the equations

$$i_X\omega_L = \mathrm{d}E_L - \mathcal{R}_L(E_L)\mathrm{d}t \quad i_X\mathrm{d}t = 1 \iff i_X\Omega_L = 0 \quad i_X\mathrm{d}t = 1 \tag{4.1}$$

have a unique solution ξ_L which is a time-dependent second-order differential equation (NSODE, for simplicity, i.e. $J\xi_L = C$ and $i_{\xi_L} \mathrm{d}t = 1$); ξ_L will be called the Euler–Lagrange vector field for L. In fact, the solutions of ξ_L are just the solutions of the Euler–Lagrange equations for L (see [15, 3, 5]).

If L is singular, we will assume that the pair $(\mathrm{d}t, \omega_L)$ is a precosymplectic structure on $\mathbb{R} \times TQ$. In this case, equations (4.1) have no solution, in general, and even if it exists it will be neither unique nor a second-order differential equation.

Remark 4.1. We note that if $L' : TQ \longrightarrow \mathbb{R}$ is a time-independent Lagrangian function and define $L : \mathbb{R} \times TQ \longrightarrow \mathbb{R}$ by $L(t, q^A, \dot{q}^A) = L'(q^A, \dot{q}^A)$, we deduce that the pair $(\mathrm{d}t, \omega_L)$ is a precosymplectic structure if and only if $\omega_{L'}$ is presymplectic, where $\omega_{L'} = -\mathrm{d}(J^*(\mathrm{d}L'))$ is the Poincaré–Cartan 2-form on TQ (see section 7).

The Legendre map $Leg : \mathbb{R} \times TQ \longrightarrow \mathbb{R} \times T^*Q$ is locally written as $Leg(t, q^A, \dot{q}^A) = (t, q^A, p_A)$, where $p_A = \partial L/\partial \dot{q}^A$ are the generalized momenta. In what follows, we will suppose that L is almost regular, i.e. $M_1 = Leg(\mathbb{R} \times TQ)$ is a submanifold of $\mathbb{R} \times T^*Q$ and $Leg : \mathbb{R} \times TQ \longrightarrow M_1$ is a submersion with connected fibres (this implies that the rank of the Hessian matrix is constant and it is equal to $\dim M_1 - (m + 1)$). The submanifold M_1 will be called the primary constraint submanifold. We have that $\ker T Leg = \ker \omega_L \cap V(\mathbb{R} \times TQ) = \ker \Omega_L \cap V(\mathbb{R} \times TQ)$, where $V(\mathbb{R} \times TQ)$ is the vertical bundle of the canonical projection $\tilde{\tau}_Q : \mathbb{R} \times TQ \longrightarrow \mathbb{R} \times Q$. Since L is almost regular, the energy E_L is constant along the fibres of Leg and, therefore, E_L projects onto a function h_1 on M_1, i.e. $h_1(Leg(x)) = E_L(x)$, for all $x \in \mathbb{R} \times TQ$ (see [5, 11]).

Denote by ω_Q the canonical symplectic form on T^*Q. Since the pair $(\mathrm{d}t, \omega_Q)$ is a cosymplectic structure on $\mathbb{R} \times T^*Q$, it defines a Poisson bracket $\{\ ,\ \}$ on $\mathbb{R} \times T^*Q$. Moreover, if $i : M_1 \longrightarrow \mathbb{R} \times T^*Q$ is the natural embedding of M_1 into $\mathbb{R} \times T^*Q$, $\omega_1 = i^*\omega_Q$ and $\eta_1 = i^*\mathrm{d}t$ then, since $Leg_1 : \mathbb{R} \times TQ \longrightarrow M_1$ is a submersion and $Leg_1^*\omega_1 = \omega_L$ and $Leg_1^*\eta_1 = \mathrm{d}t$, we deduce that (η_1, ω_1) is a precosymplectic structure on M_1. If we put $\Omega_1 = \omega_1 + \mathrm{d}h_1 \wedge \eta_1$, the Hamilton equations of the motion on M_1 are

$$i_X\Omega_1 = 0 \quad i_X\eta_1 = 1 \iff i_X\omega_1 = \mathrm{d}h_1 - \mathcal{R}_1(h_1)\eta_1 \quad i_X\eta_1 = 1 \tag{4.2}$$

where $\mathcal{R}_1$ is a vector field on M_1 such that $i_{\mathcal{R}_1}\omega_1 = 0$, $i_{\mathcal{R}_1}\eta_1 = 1$. (Note that there are many choices for $\mathcal{R}_1$). However, in general, equations (4.2) have no solution.

If k is the rank of the Hessian matrix, there exist $m - k$ independent constraints ϕ^a which describe M_1. The functions ϕ^a were called by Dirac primary constraints. If H is an arbitrary extension of h_1 to $\mathbb{R} \times T^*Q$, all the Hamiltonian functions of the form $\tilde{H} = H + \sum_a \lambda_a\phi^a$, where λ_a are Lagrange multipliers, are weakly equal, i.e. $\tilde{H}_{|M_1} = H_{|M_1} = h_1$. The Hamilton equations of the motion are written in terms of the canonical Poisson bracket on $\mathbb{R} \times T^*Q$ as follows:

$$\frac{\mathrm{d}q^A}{\mathrm{d}t} = \{q^A, \tilde{H}\} \qquad \frac{\mathrm{d}p_A}{\mathrm{d}t} = \{p_A, \tilde{H}\} \qquad \phi^a = 0\,.$$

Thus, there is an ambiguity in the description of the dynamics.

Let $\Omega_{\tilde{H}}$ be the 2-form on $\mathbb{R} \times T^*Q$ given by $\Omega_{\tilde{H}} = \omega_Q + \mathrm{d}\tilde{H} \wedge \mathrm{d}t$. A solution of the equations of motion $i_X\Omega_{\tilde{H}} = 0$, $i_X\mathrm{d}t = 1$ always exists. In fact, X is the evolution vector field $E_{\tilde{H}}$ associated with the function $\tilde{H}$ with respect to the canonical cosymplectic structure $(\mathrm{d}t, \omega_Q)$ on $\mathbb{R} \times T^*Q$. Since the constraints must be preserved in the time, i.e. $(E_{\tilde{H}})_{|M_1}$ must be tangent to M_1 we get

$$\frac{\partial\phi^b}{\partial t} + \{\phi^b, H\} + \sum_a \lambda_a\{\phi^b, \phi^a\} \approx 0 \tag{4.3}$$

i.e. $(\partial\phi^b/\partial t + \{\phi^b, H\} + \sum_a \lambda_a\{\phi^b, \phi^a\})_{|M_1} = 0$. The vanishing of these expresions can lead two kinds of consequences: some of the arbitrary functions λ_a may be determined or new constraints may arise. These new constraints are called secondary constraints. The primary and secondary constraints define the submanifold M_2. Now, we can proceed in a similar way with the secondary constraints, because they should also be preserved in time. This process may be continued and the initial problem is solvable if we arrive at some final constraint submanifold M_f where consistent solutions exist. It is possible to give a classification of the constraints generated by this algorithm in order to clarify the ambiguity of the dynamics. A constraint ϕ of M_i (the i-ary constraint submanifold) is said to be first class if $\{\phi, \phi^a\} \approx 0$ for each constraint ϕ^a of M_i. Otherwise, ϕ is said to be second class.

This constraint algorithm was globalized in [5] and [11]. In fact, equations (4.2) suggest that we should consider the following general situation.

Let (η, ω) be a precosymplectic structure on a manifold S, $\mathcal{R}$ a vector field such that $i_{\mathcal{R}}\omega = 0$, $i_{\mathcal{R}}\eta = 1$, and h a real differentiable function. Consider the points of S where the equations

$$i_X\omega = \mathrm{d}h - \mathcal{R}(h)\eta \qquad i_X\eta = 1 \tag{4.4}$$

have a solution and suppose that this set S_2 is a submanifold of S. Nevertheless, the solutions of (4.4) on S_2 are not necessarily tangent to S_2. Hence, we take the points of S_2 on which there exists a solution which is tangent to S_2. Thus, a new submanifold S_3 is obtained and the process may be continued. Note that the tangency condition is the geometrical translation of the preservation of the constraints. We have the sequence of submanifolds $\cdots \longrightarrow S_k \longrightarrow \cdots \longrightarrow S_2 \longrightarrow S_1 = S$. Alternatively, these submanifolds can be described as follows:

$$S_i = \{x \in S_{i-1}/(\mathrm{d}h + (1 - \mathcal{R}(h))\eta)_x(v) = 0 \quad \forall v \in T_x^{\perp}S_{i-1}\}$$

where

$$T_x^{\perp}S_{i-1} = \{v \in T_xS/(i_v\omega_x - \eta_x(v)\eta_x)_{|T_xS_{i-1}} = 0\}\,.$$

We call S_2 the secondary constraint submanifold, S_3 the tertiary constraint submanifold, and, in general, S_i is the i-ary constraint submanifold. If the algorithm stabilizes, i.e. there exists a positive integer k such that $S_k = S_{k+1}$ and $\dim S_k > 0$, we obtain a final submanifold S_f such that a solution X on S_f exists, i.e. X is a vector field on S_f satisfying $(i_X\omega = \mathrm{d}h - \mathcal{R}(h)\eta,\ \ i_X\eta = 1)_{|S_f}$.

This algorithm can be applied when we consider the particular system $(M_1, \eta_1, \omega_1, h_1, \mathcal{R}_1)$. Moreover, if $\mathcal{R}_L$ is a vector field on $\mathbb{R} \times TQ$ which is Leg_1-projectable on the vector field $\mathcal{R}_1$, the systems $(\mathbb{R} \times TQ, \mathrm{d}t, \omega_L, E_L, \mathcal{R}_L)$ and $(M_1, \eta_1, \omega_1, h_1, \mathcal{R}_1)$ are equivalent and they are related by the Legendre transformation (see [5, 11]).

5. Lagrangian systems with a global dynamics

Assume that the precosymplectic system $(\mathbb{R} \times TQ, \mathrm{d}t, \omega_L, E_L)$ admits a global dynamics, i.e. there exists at least a vector field ξ on $\mathbb{R} \times TQ$ such that ξ satisfies the equations of motion: $i_\xi\Omega_L = 0$, $i_\xi \mathrm{d}t = 1$. In such a case, the submanifold $M_1 = Leg(\mathbb{R} \times TQ)$ is the final constraint submanifold and there are no secondary constraints.

We denote by Φ^a, $1 \leqslant a \leqslant s$, the second-class constraints and by ϕ^i, $1 \leqslant i \leqslant p$, the first-class constraints. The matrix $\tilde{\mathcal{C}}$ whose entries are $\tilde{\mathcal{C}}^{ab} = \{\Phi^a, \Phi^b\}$ is non-singular on M_1 (see [19, 20]) and, for simplicity, we will assume that $\tilde{\mathcal{C}}$ is non-singular on $\mathbb{R} \times T^*Q$. Thus, s is even, say $s = 2r$. The determinant of the matrix $\tilde{\mathcal{C}}$ is equal to the determinant of the matrix $\mathcal{C}$ with entries $\mathcal{C}^{ab} = \{\Phi^a, \Phi^b\} + (\partial\Phi^a/\partial t)\partial\Phi^b/\partial t = (\operatorname{grad}\Phi^b)(\Phi^a)$ (see appendix A for a proof of this). Therefore, the matrix $\mathcal{C}$ is also non-singular and we will denote by $(\mathcal{C}_{ab})$ its inverse matrix.

Since $\{\phi^i, \Phi^a\} \approx 0$ and $\{\phi^i, \phi^j\} \approx 0$ for all $1 \leqslant j \leqslant p$ and $1 \leqslant a \leqslant s$, we have that $(X_{\phi^i})_{|M_1}$ is tangent to M_1 for all $1 \leqslant i \leqslant p$. Consequently, if $\tilde{\mathcal{R}}_1$ is a vector field on M_1 such that $i_{\tilde{\mathcal{R}}_1}\omega_1 = 0$ and $\eta_1(\tilde{\mathcal{R}}_1) = 1$, we deduce that

$$0 = (i_{\tilde{\mathcal{R}}_1}\omega_1)((X_{\phi^i})_{|M_1}) = -(i_{X_{\phi^i}}\omega_Q)_{|M_1}(\tilde{\mathcal{R}}_1) = -(\mathrm{d}\phi^i)_{|M_1}(\tilde{\mathcal{R}}_1)$$
$$+ \left(\frac{\partial\phi^i}{\partial t}\right)_{|M_1} \mathrm{d}t_{|M_1}(\tilde{\mathcal{R}}_1) = \left(\frac{\partial\phi^i}{\partial t}\right)_{|M_1} \eta_1(\tilde{\mathcal{R}}_1) = \left(\frac{\partial\phi^i}{\partial t}\right)_{|M_1} . \tag{5.1}$$

In particular, $(\operatorname{grad}\phi^i)_{|M_1} = (X_{\phi^i})_{|M_1}$ for all i.

Now, let D $(\bar{D})$ be the distribution on $\mathbb{R} \times T^*Q$ generated by the vector fields $\operatorname{grad}\Phi^a$ ($\operatorname{grad}\Phi^a$ and $\operatorname{grad}\phi^i$). Denote by $D^\perp(x)$ $(\bar{D}^\perp(x))$ the orthocomplement of the subspace $D(x)$ $(\bar{D}(x))$ in the cosymplectic vector space $(T_x(\mathbb{R} \times T^*Q), \mathrm{d}t(x), \omega_Q(x))$ for all $x \in \mathbb{R} \times T^*Q$. If x_1 is a point of M_1, from equations (2.1), (2.2), (2.3), (5.1) and proposition 2.2, we obtain that $\bar{D}^\perp(x_1) = T_{x_1}M_1$ and $T^\perp_{x_1}M_1 = (\bar{D}^\perp)^\perp(x_1) = \big\langle\big(X_{\Phi^a} - (\partial\Phi^a/\partial t)\partial/\partial t\big)(x_1), X_{\phi^i}(x_1)\big\rangle$. Moreover, if x is a point of $\mathbb{R} \times T^*Q$, since the matrix $\mathcal{C}$ is non-singular, we have that $D(x) \cap D^\perp(x) = \{0\}$ and, thus, from proposition 2.2, we conclude that $T_x(\mathbb{R} \times T^*Q) = D(x) \oplus D^\perp(x)$.

Let $\mathcal{Q} : D \oplus D^\perp \longrightarrow D$ be the projection on D along $D^\perp$, and put $\mathcal{P} = \mathrm{id} - \mathcal{Q}$. The projector $\mathcal{Q}$ is explicitly given by

$$\mathcal{Q} = \sum_{a,b} \mathcal{C}_{ab} \operatorname{grad}\Phi^a \otimes \mathrm{d}\Phi^b . \tag{5.2}$$

Using the fact that the pair (η_1, ω_1) is a precosymplectic structure on M_1 and the fact that the matrix $\mathcal{C}$ is regular, we deduce that

$$\ker\omega_1 \cap \ker\eta_1 = T^\perp M_1 \cap TM_1 = (\bar{D}^\perp)^\perp_{|M_1} \cap TM_1 = \langle(X_{\phi^i})_{|M_1}\rangle . \tag{5.3}$$

Next, we consider on $\mathbb{R} \times T^*Q$ the 1-form η and the vector field $\mathcal{R}$ defined by $\eta = \mathcal{P}^*(\mathrm{d}t)$ and $\mathcal{R} = \mathcal{P}(\partial/\partial t)$. It is clear that $\mathcal{R}(\Phi^b) = 0$ for all b. Moreover, using equations (5.1) and (5.2), we obtain that $\mathcal{R}_{|M_1}(\phi^i) = 0$ for all i and, therefore the restriction $\mathcal{R}_1$ of $\mathcal{R}$ to M_1 is tangent to M_1.

A direct computation using the definition of $D^\perp$ (see equation (2.1)) and the fact that $i_{\frac{\partial}{\partial t}}\omega_Q = 0$ proves that $\mathcal{P}^*(i_{\mathcal{R}}\omega_Q) = (1 - \mathrm{d}t(\mathcal{R}))\eta$. In particular, $\mathrm{d}t(\mathcal{R}) = 1$ or $\mathrm{d}t(\mathcal{R}) = 0$. Now, if $\mathrm{d}t(\mathcal{R}) = 0$, we get $\mathcal{P}^*(i_{\mathcal{R}}\omega_Q) = \eta$. But it is not possible since $\mathcal{R}_1$ is tangent to M_1 and there exists a vector field $\tilde{\mathcal{R}}_1$ on M_1 such that $i_{\tilde{\mathcal{R}}_1}\omega_1 = 0$, $i_{\tilde{\mathcal{R}}_1}\eta_1 = 1$. Thus

$$\mathrm{d}t(\mathcal{R}) = 1 \tag{5.4}$$

which implies that

$$i_{\mathcal{R}}\omega_Q = \mathcal{Q}^*(\mathrm{d}t) . \tag{5.5}$$

From equations (5.4) and (5.5), we obtain that $i_{\mathcal{R}_1}\omega_1 = 0$ and $i_{\mathcal{R}_1}\eta_1 = 1$.

Let ω be the 2-form on $\mathbb{R} \times T^*Q$ defined by $\omega = \bar{\mathcal{P}}^*\omega_Q$, where $\bar{\mathcal{P}} = \mathcal{P} - \eta \otimes \mathcal{R}$. Using remark 3.4, and equations (5.4) and (5.5), we have the following proposition.

Proposition 5.1. (i) The pair (η, ω) is an almost precosymplectic structure on $\mathbb{R} \times T^*Q$ and $i_{\mathcal{R}}\omega = 0$, $\eta(\mathcal{R}) = 1$. (ii) The almost product structure $(\mathcal{P}, \mathcal{Q})$ on $\mathbb{R} \times T^*Q$ is adapted to the triple $(\eta, \omega, \mathcal{R})$.

Now, using proposition 5.1, we can define the corresponding bracket of functions $\{\ ,\ \}_{\mathrm{D}}$ on $\mathbb{R} \times T^*Q$. In order to obtain an expression for the bracket $\{\ ,\ \}_{\mathrm{D}}$ we will first show that

$$\bar{\mathcal{P}}(X_F) = X_{(F,\mathcal{P})} \tag{5.6}$$

for all $F \in C^\infty(\mathbb{R} \times T^*Q)$. In fact, from equations (5.4) and (5.5), we have that

$$i_{\bar{\mathcal{P}}X_F}\omega = \mathcal{P}^*\left(\mathrm{d}F - \left(\frac{\partial F}{\partial t} - \mathcal{Q}^*(\mathrm{d}t)(X_F)\right)\mathrm{d}t\right) = \mathcal{P}^*(\mathrm{d}F - \mathcal{R}(F)\eta) \qquad i_{\bar{\mathcal{P}}X_F}\eta = 0 \tag{5.7}$$

which proves (5.7). Consequently, using equations (5.5) and (5.7), we conclude that

$$\{F, G\}_D = \omega(X_{(F,\mathcal{P})}, X_{(G,\mathcal{P})}) = \{F, G\} - \sum_{a,b}\mathcal{C}_{ab}\{\Phi^b, G\}\{F, \Phi^a\}$$

$$- \sum_{a,b,a',b'}\mathcal{C}_{ab}\mathcal{C}_{a'b'}\{\Phi^b, F\}\{\Phi^{b'}, G\}\frac{\partial\Phi^a}{\partial t}\frac{\partial\Phi^{a'}}{\partial t}$$

for all $F, G \in C^\infty(\mathbb{R}\times T^*Q)$. From equation (5.4), we obtain that $\bar{\mathcal{P}}X_{\Phi^a} = 0$, for $1 \leqslant a \leqslant s$, which implies that $\{F, \Phi^a\}_{\mathrm{D}} = 0$ for $F \in C^\infty(\mathbb{R} \times T^*Q)$ (see equation (5.7)). Thus, the second-class constraints are Casimir functions for the bracket $\{\ ,\ \}_{\mathrm{D}}$. Furthermore, if ϕ is a constraint function then, it is clear that $\{\phi, \phi^i\}_{\mathrm{D}} \approx 0$, for $1 \leqslant i \leqslant p$.

The bracket $\{\ ,\ \}_D$ is called the time-dependent Dirac bracket on $\mathbb{R} \times T^*Q$. Note that if $F, G \in C^\infty(T^*Q)$ and the constraints Φ^a do not depend of t then $\{F, G\}_{\mathrm{D}}$ is the usual Dirac bracket on T^*Q. Moreover, if $\bar{\mathcal{Q}} = \mathcal{Q} + \eta \otimes \mathcal{R}$ and the almost product structure $(\bar{\mathcal{P}}, \bar{\mathcal{Q}})$ is integrable, then the 2-form ω is closed and, from proposition 3.5, we deduce that $\{\ ,\ \}_{\mathrm{D}}$ is actually a Poisson bracket on $\mathbb{R} \times T^*Q$.

Let H be an arbitrary extension of h_1 to $\mathbb{R} \times T^*Q$. We have the following theorem.

Theorem 5.2. If E_H is the evolution vector field associated with H with respect to the canonical cosymplectic structure $(\mathrm{d}t, \omega_Q)$ on $\mathbb{R}\times T^*Q$ and $E_{(H,\mathcal{P})}$ is the $\mathcal{P}$-evolution vector field associated with H, then

$$E_{(H,\mathcal{P})} = \mathcal{R} + \bar{\mathcal{P}}(X_H) = (1 - \eta(X_H))\mathcal{R} + \mathcal{P}(X_H) \tag{5.8}$$

and the restriction of $E_{(H,\mathcal{P})}$ to M_1 is tangent to M_1 and it is a solution of the equations of motion.

Proof. From equations (3.3) and (5.7), we obtain equation (5.8) directly.

Next, from (5.4) we have

$$\mathrm{d}t(E_{(H,\mathcal{P})}) = 1\,. \tag{5.9}$$

Furthermore, if X is a vector field on M_1, $(i_{E_{(H,\mathcal{P})}}\bar{\mathcal{Q}}^*\omega_Q)_{|M_1}(X) = 0$ and, using equations (3.2) and (3.3), we get

$$(i_{E_{(H,\mathcal{P})}}\omega_Q)_{|M_1}(X) = (i_{E_{(H,\mathcal{P})}}\omega)_{|M_1}(X) = (\mathrm{d}h_1 - \mathcal{R}_1(h_1)\eta_1)(X)\,. \tag{5.10}$$

Let ξ_1 be a global solution of the equations of motion. Using equations (5.9) and (5.10), we obtain that

$$(E_{(H,\mathcal{P})} - \xi_1)(x_1) \in T^\perp_{x_1}M_1 = (\bar{D}^\perp)^\perp(x_1) = \left\langle\left(X_{\Phi^a} - \frac{\partial\Phi^a}{\partial t}\frac{\partial}{\partial t}\right)(x_1), (X_{\phi^i})(x_1)\right\rangle$$

for all $x_1 \in M_1$. Now, since $(E_{(H,\mathcal{P})})_{|M_1}(\Phi^b) = \xi_1(\Phi^b) = 0$, for all b, we conclude that $(E_{(H,\mathcal{P})} - \xi_1)(x_1) \in \langle(X_{\phi^i})(x_1)\rangle \subseteq T_{x_1}M_1$ for all $x_1 \in M_1$. This shows that $(E_{(H,\mathcal{P})})_{|M_1}$ is tangent to M_1 which concludes the proof of our result. □

If $F \in C^\infty(\mathbb{R} \times T^*Q)$ then, using that $E_{(H,\mathcal{P})}(F) = \mathcal{R}(F) + \{F, H\}_{\mathrm{D}}$, we deduce that the evolution of the observable $f = F_{|M_1}$ is given by $\dot{f} = \mathcal{R}_1(f) + (\{F, H\}_{\mathrm{D}})_{|M_1}$.

From equations (5.2), (5.4) and (5.8), we also have that

$$E_{(H,\mathcal{P})} = \frac{\partial}{\partial t} + X_H - \sum_{a,b} \mathcal{C}_{ab}\{\Phi^b, H\} X_{\Phi^a} - \sum_{a,b} \mathcal{C}_{ab} \frac{\partial \Phi^b}{\partial t}\left(1 + \sum_{a',b'} \mathcal{C}_{a'b'}\{\Phi^{b'}, H\} \frac{\partial \Phi^{a'}}{\partial t}\right) X_{\Phi^a}\,.$$

By a straightforward computation we obtain the result that $(E_{(H,\mathcal{P})})_{|M_1}$ is precisely the vector field $(E_{\tilde{H}})_{|M_1}$ where $\tilde{H}$ is the Hamiltonian function defined from (4.3), i.e.

$$\tilde{H} = H - \sum_{a,b} \tilde{\mathcal{C}}_{ab} \left(\frac{\partial \Phi^b}{\partial t} + \{\Phi^b, H\}\right) \Phi^a\,.$$

Now, in order to fix the gauge, we consider an almost product structure $(\mathcal{A}_1, \mathcal{B}_1)$ adapted to the triple $(\eta_1, \omega_1, \mathcal{R}_1)$. Thus, if ξ_1 is a solution of the equations of motion on M_1, we can select a unique solution $\mathcal{A}_1(\xi_1)$ such that $\mathcal{A}_1(\xi_1) \in Im\mathcal{A}_1$. In particular, if H is an arbitrary extension of h_1, $\xi_1 = (E_{(H,\mathcal{P})})_{|M_1}$ is a solution of the equations of motion on M_1, and we fix the gauge by taking $\mathcal{A}_1(\xi_1) = \mathcal{A}_1((E_{(H,\mathcal{P})})_{|M_1}) = \mathcal{R}_1 + \mathcal{A}_1(\bar{\mathcal{P}}(X_H)_{|M_1})$ (see equation (5.8)). We also have the corresponding bracket of functions $\{\ ,\ \}_{\mathcal{A}_1}$ on M_1 which is a Poisson bracket if the almost product structure $(\bar{\mathcal{A}}_1, \bar{\mathcal{B}}_1)$ is integrable, where $\bar{\mathcal{A}}_1 = \mathcal{A}_1 - \eta_1 \otimes \mathcal{R}_1$ and $\bar{\mathcal{B}}_1 = \mathcal{B}_1 + \eta_1 \otimes \mathcal{R}_1$.

The above results are summarized in table 1.

Table 1. First- and second-class primary constraints.

$\mathbb{R} \times T^*Q$	$\mathbb{R} \times T^*Q$	M_1
$\left(\mathrm{d}t, \omega_Q, \frac{\partial}{\partial t}\right)$	$(\eta, \omega, \mathcal{R})$	$(\eta_1, \omega_1, \mathcal{R}_1)$
$\{\ ,\ \}$	$\{\ ,\ \}_D$	$\{\ ,\ \}_{\mathcal{A}_1}$
	$(\mathcal{P}, \mathcal{Q})$	$(\mathcal{A}_1, \mathcal{B}_1)$

Next, we will study two particular cases:

1. All the primary constraints are second class. In this case, using equation (5.3) and proposition 2.1, we deduce that the pair (η_1, ω_1) is a cosymplectic structure on M_1 and $\mathcal{R}_1$ is the Reeb vector field. Moreover, there exists a unique solution ξ_1 of the equations of motion $i_{\xi_1}\omega_1 = \mathrm{d}h_1 - \mathcal{R}_1(h_1)\eta_1$, $i_{\xi_1}\eta_1 = 1$. In fact, ξ_1 is the evolution vector for the function $h_1 : M_1 \longrightarrow \mathbb{R}$ with respect to the cosymplectic structure (η_1, ω_1). On the other hand, if the almost product structure $(\bar{\mathcal{P}}, \bar{\mathcal{Q}})$ is integrable and on M_1 we consider the Poisson bracket $\{\ ,\ \}_1$ defined by the cosymplectic structure (η_1, ω_1), and on $\mathbb{R} \times T^*Q$ the Dirac bracket $\{\ ,\ \}_D$, which is also a Poisson bracket, we get that the canonical embedding $i : M_1 \longrightarrow \mathbb{R} \times T^*Q$ is a Poisson morphism. This result follows using the fact that if F is a C^∞-function on $\mathbb{R} \times T^*Q$ and f is the restriction of F to M_1 then $(X_{(F,\mathcal{P})})_{|M_1}$ is the Hamiltonian vector field X_f with respect to the cosymplectic structure (η_1, ω_1) (see the proof of theorem 5.2).

2. All the primary constraints are first class. In this case, $\eta = \mathrm{d}t$, $\omega = \omega_Q$, $\mathcal{R} = \partial/\partial t$, and, if H is an arbitrary extension to $\mathbb{R} \times T^*Q$ of h_1, then $\xi_1 = (E_{(H,\mathcal{P})})_{|M_1} = (E_H)_{|M_1}$ is tangent to M_1, and it is a solution of the equations of motion. Furthermore, the classical procedure consists in choosing functions f^j, $1 \leqslant j \leqslant p$, on T^*Q such that the matrix $\{f^i, \phi^j\} = (c^{ij})$ is regular. The determinant of this matrix is called the Faddev–Popov determinant. Then, if H is an arbitrary extension of $h_1 : M_1 \to \mathbb{R}$, $\tilde{H} = H + \sum_i \lambda_i \phi^i$ and we impose the tangency of the evolution vector fields of the Hamiltonian functions $\tilde{H}$ to the submanifold defined by the constraints $\{f^j\}$, we get that $\lambda_i \approx (\sum_j c_{ij}\{H, f^j\})$, (c_{ij}) being the inverse matrix of (c^{ij}). Thus, we have fixed the gauge. But the above construction is equivalent to take an almost product structure $(\mathcal{A}, \mathcal{B})$ on $\mathbb{R} \times T^*Q$ given by $\mathcal{B} = \sum_{i,j} c_{ij} X_{\phi^i} \otimes \mathrm{d}f^j$ and $\mathcal{A} = \mathrm{id} - \mathcal{B}$. The almost product structure $(\mathcal{A}, \mathcal{B})$ restricts on M_1 and its restriction $(\mathcal{A}_1 = \mathcal{A}_{|M_1}, \mathcal{B}_1 = \mathcal{B}_{|M_1})$ is adapted to the triple $(\eta_1, \omega_1, \mathcal{R}_1)$. Moreover, the solution of the equations of motion on M_1 obtained using the classical procedure is just the $\mathcal{A}_1$-evolution vector field associated with the function $h_1 : M_1 \to \mathbb{R}$.

6. Lagrangian systems with secondary constraints

We denote by Φ^a, $1 \leqslant a \leqslant s$, the primary second-class constraints and by ϕ^i, $1 \leqslant i \leqslant p$, the primary first-class constraints.

By applying the constraint algorithm to the precosymplectic system $(M_1, \eta_1, \omega_1, h_1)$ we obtain a sequence of submanifolds

$$M_f \longrightarrow \cdots \longrightarrow M_k \longrightarrow M_{k-1} \longrightarrow \cdots \longrightarrow M_2 \longrightarrow M_1 \longrightarrow \mathbb{R} \times T^*Q\,.$$

Assume that the algorithm stabilizes, i.e. there exists a positive integer k such that $M_{k+1} = M_k$ and $\dim M_k > 0$. Suppose that M_f is the final constraint submanifold. We will call each constraint which is not primary a secondary constraint. The final constraint submanifold M_f will be determined by all the primary and secondary constraints. We denote by $\bar{\Phi}^b$, $1 \leqslant b \leqslant \bar{s}$, the secondary second-class constraints and by $\bar{\phi}^j$, $1 \leqslant j \leqslant \bar{p}$, the secondary first-class constraints. The primary second-class constraints of M_1 are also second class on M_f but the primary first-class constraints may be first or second class on M_f. Then we can suppose that $\phi^{i'}$, $1 \leqslant i' \leqslant p'$, are primary first-class constraints which are also first class on M_f and $\phi^{i''}$, $1 \leqslant i'' \leqslant p''$, are primary first-class constraints which are second class on M_f, where $p' + p'' = p$. As in section 5, we will assume that there exists at least a vector field $\tilde{\mathcal{R}}_f$ on M_f such that $i_{\tilde{\mathcal{R}}_f}(\omega_1)_{|M_f} = 0$ and $(\eta_1)_{|M_f}(\tilde{\mathcal{R}}_f) = 1$ (note that in section 5 $M_f = M_1$).

We denote by $\{\chi^\alpha\}$ the set of all the second-class constraints on M_f. Then, the pair $(\mathcal{P}, \mathcal{Q})$ is an almost product structure on $\mathbb{R} \times T^*Q$, where

$$\mathcal{Q} = \sum_{\alpha,\beta} \mathcal{C}_{\alpha\beta}\, \mathrm{grad}\, \chi^\alpha \otimes \mathrm{d}\chi^\beta \qquad \mathcal{P} = \mathrm{id} - \mathcal{Q}$$

$(\mathcal{C}_{\alpha\beta})$ being the inverse matrix of $\big(\{\chi^\alpha, \chi^\beta\} + (\partial\chi^\alpha/\partial t)\partial\chi^\beta/\partial t\big)$. The restriction $\mathcal{R}_1$ $(\mathcal{R}_f)$ of the vector field $\mathcal{R} = \mathcal{P}(\partial/\partial t)$ to M_1 (M_f) is tangent to M_1 (M_f) and the almost product structure $(\mathcal{P}, \mathcal{Q})$ is adapted to the triple $(\eta, \omega, \mathcal{R})$, with $\eta = \mathcal{P}^*(\mathrm{d}t)$, $\omega = \bar{\mathcal{P}}^*(\omega_Q)$ and $\bar{\mathcal{P}} = \mathcal{P} - \eta \otimes \mathcal{R}$. The vector field $\mathcal{R}_f$ satisfies the relations: $i_{\mathcal{R}_f}(\omega_1)_{|M_f} = 0$, $(\eta_1)_{|M_f}(\mathcal{R}_f) = 1$. As in section 5, we also define a Dirac bracket by

$$\begin{aligned}\{F, G\}_D &= \omega(X_{(F,\mathcal{P})}, X_{(G,\mathcal{P})}) \\ &= \{F, G\} - \sum_{\alpha,\beta} \mathcal{C}_{\alpha\beta}\{\chi^\beta, G\}\{F, \chi^\alpha\} - \sum_{\alpha,\beta,\alpha',\beta'} \mathcal{C}_{\alpha\beta}\mathcal{C}_{\alpha'\beta'}\{\chi^\beta, F\}\{\chi^{\beta'}, G\}\frac{\partial\chi^\alpha}{\partial t}\frac{\partial\chi^{\alpha'}}{\partial t}\end{aligned}$$

for all $F, G \in C^\infty(\mathbb{R} \times T^*Q)$.

Now, if H is an arbitrary extension of h_1 to $\mathbb{R} \times T^*Q$, the restriction to M_f of the $\mathcal{P}$-evolution vector field $E_{(H,\mathcal{P})} = \mathcal{R} + \bar{\mathcal{P}}(X_H)$ is tangent to M_f and is a solution of the equations of motion

$$(i_{(E_{(H,\mathcal{P})})}\omega_1 = \mathrm{d}h_1 - \mathcal{R}_1(h_1)\eta_1, \quad i_{(E_{(H,\mathcal{P})})}\eta_1 = 1)_{|M_f} .$$

In order to fix the gauge, we consider an almost product structure $((\mathcal{A}_1)_f, (\mathcal{B}_1)_f)$ on M_f which is adapted to the distribution $\bar{D}_f = \ker\omega_1 \cap \ker\eta_1 \cap TM_f$, i.e. $\ker(\mathcal{A}_1)_f = \bar{D}_f$ and such that $(\mathcal{A}_1)_f(\mathcal{R}_f) = \mathcal{R}_f$. Then, we fix the gauge by taking $(\mathcal{A}_1)_f((E_{(H,\mathcal{P})})_{|M_f}) = \mathcal{R}_f + (\mathcal{A}_1)_f((\bar{\mathcal{P}}X_H)_{|M_f})$.

Next, suppose that we apply the constraint algorithm to the precosymplectic system $(\mathbb{R} \times TQ, \mathrm{d}t, \omega_L, E_L)$ in order to solve the equations of motion (4.1). We obtain a final constraint submanifold P_f in such a way that the restriction to P_f of the submersion $Leg_1 : \mathbb{R} \times TQ \longrightarrow M_1$ is a surjective submersion $Leg_f : P_f \longrightarrow M_f$ of P_f onto M_f. Moreover, if ξ_f is a solution of the equations of motion on P_f which is Leg_f-projectable onto a vector field $(\xi_1)_f$ in M_f, then $(\xi_1)_f$ is a solution of the equations of motion on M_f and, conversely (see [5, 11]).

Now, let $\mathcal{R}_L$ be a vector field on $\mathbb{R} \times TQ$ which is Leg_1-projectable onto $\mathcal{R}_1$. Since the distribution $\ker TLeg_1$ restricts to P_f (see [5, 11]) we deduce that the restriction $(\mathcal{R}_L)_f$ to P_f of $\mathcal{R}_L$ is tangent to P_f, $i_{(\mathcal{R}_L)_f}(\omega_L)_{|P_f} = 0$ and $i_{(\mathcal{R}_L)_f}(\mathrm{d}t)_{|P_f} = 1$.

Consider an almost product structure $(\tilde{\mathcal{A}}_f, \tilde{\mathcal{B}}_f)$ on P_f adapted to the distribution $\tilde{D}_f = \ker\omega_L \cap \ker \mathrm{d}t \cap TP_f$ and such that it is Leg_f-projectable onto an almost product structure $((\mathcal{A}_1)_f, (\mathcal{B}_1)_f)$ on M_f, and $(\tilde{\mathcal{A}}_f)((\mathcal{R}_L)_f) = (\mathcal{R}_L)_f$ (note that such an almost product structure always exists). Then, the almost product structure $((\mathcal{A}_1)_f, (\mathcal{B}_1)_f)$ is adapted to the distribution $\bar{D}_f$, and $(\mathcal{A}_1)_f(\mathcal{R}_f) = \mathcal{R}_f$. Furthermore, if H is an extension to $\mathbb{R} \times T^*Q$ of h_1 and $\tilde{X}_f$ is a vector field on P_f which is Leg_f-projectable onto $(X_{(H,\mathcal{P})})_{|M_f} = (\bar{\mathcal{P}}X_H)_{|M_f}$, we can select a unique solution $\xi = \tilde{\mathcal{A}}_f((\mathcal{R}_L)_f + \tilde{X}_f) = (\mathcal{R}_L)_f + \tilde{\mathcal{A}}_f(\tilde{X}_f) \in Im\tilde{\mathcal{A}}_f$ of the equations of motion

$$(i_\xi\omega_L = \mathrm{d}E_L - \mathcal{R}_L(E_L)\mathrm{d}t, \quad i_\xi \mathrm{d}t = 1)_{|P_f}$$

and ξ is Leg_f-projectable onto the solution $\xi' = \mathcal{R}_f + (\mathcal{A}_1)_f((\bar{\mathcal{P}}X_H)_{|M_f}) = (\mathcal{A}_1)_f((E_{(H,\mathcal{P})})_{|M_f})$.

Remark 6.1. In general, the solution ξ does not satisfy the NSODE condition on P_f, i.e. $(J\xi = C)_{|P_f}$. However, there exists a smooth section $\alpha : M_f \longrightarrow P_f$ of the submersion $Leg_f : P_f \longrightarrow M_f$ and a unique vector field ξ_S on the submanifold $S = \alpha(M_f)$ such that (see [5, 11]):

(i) ξ_S is a solution of the equations of motion:

$$\left(i_{\xi_S}\omega_L = \mathrm{d}E_L - \mathcal{R}_L(E_L)\mathrm{d}t, \quad i_{\xi_S}\mathrm{d}t = 1\right)_{|S} \tag{6.1}$$

(ii) ξ_S verifies the NSODE condition:

$$(J(\xi_S) = C)_{|S} .$$

If we transport via α the almost product structure $((\mathcal{A}_1)_f, (\mathcal{B}_1)_f)$ to S, we obtain an almost product structure $(\mathcal{A}_S, \mathcal{B}_S)$ on S, which is adapted to $\ker\omega_L \cap \ker \mathrm{d}t \cap TS$ and such that the projection by $\mathcal{A}_S$ of any solution X of the equations of motion (6.1) is also a solution of these equations; moreover, $\mathcal{A}_S(X) = \xi_S$, and, then, it verifies the NSODE condition.

Remark 6.2. Since the map $(TLeg_f)_{|\tilde{D}_f} : \tilde{D}_f \longrightarrow \bar{D}_f$ is surjective and $\ker TLeg_f = (\ker TLeg_1)_{|P_f} \subseteq \tilde{D}_f$, we deduce that

$$\dim \tilde{D}_f = \dim \bar{D}_f + \dim \ker TLeg_1 = \dim \bar{D}_f + (m - k)$$

where k is the rank of the Hessian matrix.

Table 2 summarizes the results of this section.

Table 2. Primary and secondary constraints.

$\mathbb{R} \times TQ$: $(\mathrm{d}t, \omega_L, \mathcal{R}_L)$	$\mathbb{R} \times T^*Q$: $\left(\mathrm{d}t, \omega_Q, \dfrac{\partial}{\partial t}\right)$, $\{\ ,\ \}$ — $(\eta, \omega, \mathcal{R})$, $(\mathcal{P}, \mathcal{Q})$, $\{\ ,\ \}_D$
P_f: $(\tilde{\mathcal{A}}_f, \tilde{\mathcal{B}}_f)$, $\ker \tilde{\mathcal{A}}_f = \ker \omega_L \cap \ker \mathrm{d}t \cap TP_f$, $\tilde{\mathcal{A}}_f(\mathcal{R}_L)_f = (\mathcal{R}_L)_f$, $\{\ ,\ \}_{\tilde{\mathcal{A}}_f}$	M_f: $((\mathcal{A}_1)_f, (\mathcal{B}_1)_f)$, $\ker(\mathcal{A}_1)_f = \ker \omega_1 \cap \ker \eta_1 \cap TM_f$, $(\mathcal{A}_1)_f(\mathcal{R}_f) = \mathcal{R}_f$, $\{\ ,\ \}_{(\mathcal{A}_1)_f}$
S: $(\mathcal{A}_S, \mathcal{B}_S)$, $\{\ ,\ \}_{\mathcal{A}_S}$, ξ_S	

Example 6.3. Consider the Lagrangian function $L : \mathbb{R} \times T\mathbb{R}^3 \longrightarrow \mathbb{R}$ defined by

$$L(t, q^A, \dot{q}^A) = \tfrac{1}{2}(\dot{q}^1 + \dot{q}^2)^2 + \tfrac{1}{2}\left((q^1)^2 + (q^2)^2\right) - tq^1 .$$

Since $\mathrm{d}t \wedge \omega_L \neq 0$ and $\omega_L^2 = 0$, it follows that $(\mathrm{d}t, \omega_L)$ is a precosymplectic structure. We have that

$$M_1 = \{(t, q^A, p_A) \in \mathbb{R} \times T^*\mathbb{R}^3 \ / \ p_1 - p_2 = 0\ ,\ \ p_3 = 0\}$$

and the primary constraints are thus $\phi^1 = p_1 - p_2$ and $\phi^2 = p_3$. Both are first class.

If we take cordinates (t, q^1, q^2, q^3, p_1) on M_1, we get

$$\omega_1 = i^*\omega_Q = \mathrm{d}q^1 \wedge \mathrm{d}p_1 + \mathrm{d}q^2 \wedge \mathrm{d}p_1 \qquad \eta_1 = \mathrm{d}t$$

and, then

$$\ker \omega_1 \cap \ker \eta_1 = \langle (X_{\phi^1})_{|M_1}, (X_{\phi^2})_{|M_1} \rangle = \left\langle \frac{\partial}{\partial q^1} - \frac{\partial}{\partial q^2}, \frac{\partial}{\partial q^3} \right\rangle .$$

Since the Hamiltonian function $h_1 : M_1 \longrightarrow \mathbb{R}$ is

$$h_1 = \tfrac{1}{2}(p_1)^2 + \tfrac{1}{2}\left((q^1)^2 + (q^2)^2\right) - tq^1$$

a new (secondary) constraint $\phi^3 = t - q^1 + q^2$ arises and, therefore, the secondary constraint submanifold is given by

$$M_2 = \{(t, q^A, p_A) \in \mathbb{R} \times T^*\mathbb{R}^3 \ / \ p_1 - p_2 = 0\ ,\ \ p_3 = 0\ ,\ \ t - q^1 + q^2 = 0\} .$$

M_2 is, moreover, the final constraint submanifold, i.e. there are no tertiary constraints.

We see that ϕ^1 and ϕ^3 are second class on M_2 but ϕ^2 is first class on M_2.

We construct an almost product structure $(\mathcal{P}, \mathcal{Q})$ on $\mathbb{R} \times T^*\mathbb{R}^3$ by setting

$$\begin{aligned}\mathcal{Q} = \frac{1}{4}\left(\frac{\partial}{\partial q^1} \otimes \mathrm{d}p_1 - \frac{\partial}{\partial q^1} \otimes \mathrm{d}p_2 - \frac{\partial}{\partial q^2} \otimes \mathrm{d}p_1 + \frac{\partial}{\partial q^2} \otimes \mathrm{d}p_2\right) - \frac{1}{2}\left(\frac{\partial}{\partial q^1} \otimes \mathrm{d}t\right.\\ \left. - \frac{\partial}{\partial q^1} \otimes \mathrm{d}q^1 + \frac{\partial}{\partial q^1} \otimes \mathrm{d}q^2 - \frac{\partial}{\partial q^2} \otimes \mathrm{d}t + \frac{\partial}{\partial q^2} \otimes \mathrm{d}q^1 - \frac{\partial}{\partial q^2} \otimes \mathrm{d}q^2\right)\\ + \frac{1}{2}\left(\frac{\partial}{\partial t} \otimes \mathrm{d}p_1 - \frac{\partial}{\partial t} \otimes \mathrm{d}p_2 + \frac{\partial}{\partial p_1} \otimes \mathrm{d}p_1 - \frac{\partial}{\partial p_1} \otimes \mathrm{d}p_2\right.\\ \left. - \frac{\partial}{\partial p_2} \otimes \mathrm{d}p_1 + \frac{\partial}{\partial p_2} \otimes \mathrm{d}p_2\right)\end{aligned}$$

and $\mathcal{P} = \mathrm{id} - \mathcal{Q}$. The Dirac bracket on $\mathbb{R} \times T^*\mathbb{R}^3$ is given by

$$\{q^1, p_1\}_D = \{q^1, p_2\}_D = \{q^2, p_1\}_D = \{q^2, p_2\}_D = \tfrac{1}{2} \qquad \{q^3, p_3\}_D = 1$$

the other brackets being zero. We also have that

$$\mathcal{R} = \mathcal{P}\left(\frac{\partial}{\partial t}\right) = \frac{\partial}{\partial t} + \frac{1}{2}\left(\frac{\partial}{\partial q^1} - \frac{\partial}{\partial q^2}\right) \qquad \eta = \mathcal{P}^*(\mathrm{d}t) = \mathrm{d}t - \frac{1}{2}(\mathrm{d}p_1 - \mathrm{d}p_2)\,.$$

Observe that, as we have proved, $\mathcal{R}_2 = \mathcal{R}_{|M_2} \in \mathfrak{X}(M_2)$.

Now, let

$$H(t, q^A, p_A) = \tfrac{1}{2}(p_1)^2 + \tfrac{1}{2}\left((q^1)^2 + (q^2)^2\right) - tq^1 + \lambda\phi^1 + \mu\phi^2$$

be an arbitrary extension of h_1. We get

$$(X_H)_{|M_2} = p_1\frac{\partial}{\partial q^1} + (t - q^1)\frac{\partial}{\partial p_1} - q^2\frac{\partial}{\partial p_2} + \lambda\left(\frac{\partial}{\partial q^1} - \frac{\partial}{\partial q^2}\right) + \mu\frac{\partial}{\partial q^3}$$

and

$$\bar{\mathcal{P}}((X_H)_{|M_2}) = \frac{p_1}{2}\left(\frac{\partial}{\partial q^1} + \frac{\partial}{\partial q^2}\right) + \mu\frac{\partial}{\partial q^3} - q^2\left(\frac{\partial}{\partial p_1} + \frac{\partial}{\partial p_2}\right)$$

$$(E_{(H,\mathcal{P})})_{|M_2} = \frac{\partial}{\partial t} + \frac{p_1 + 1}{2}\frac{\partial}{\partial q^1} + \frac{p_1 - 1}{2}\frac{\partial}{\partial q^2} + \mu\frac{\partial}{\partial q^3} - q^2\left(\frac{\partial}{\partial p_1} + \frac{\partial}{\partial p_2}\right)\,.$$

We fix the gauge by taking an almost product structure $(\mathcal{A}_2, \mathcal{B}_2)$ on M_2 adapted to the distribution

$$\ker\omega_1 \cap \ker\eta_1 \cap TM_2 = \left\langle \frac{\partial}{\partial q^3}_{|M_2} \right\rangle\,.$$

For instance, we can take

$$\mathcal{A}_2(\mathcal{R}_2) = \mathcal{R}_2\,,\ \mathcal{A}_2\left(\frac{\partial}{\partial q^1} + \frac{\partial}{\partial q^2}\right) = \frac{\partial}{\partial q^1} + \frac{\partial}{\partial q^2}$$

$$\mathcal{A}_2\left(\frac{\partial}{\partial p_1} + \frac{\partial}{\partial p_2}\right) = \frac{\partial}{\partial p_1} + \frac{\partial}{\partial p_2} \qquad \mathcal{A}_2\left(\frac{\partial}{\partial q^3}\right) = 0$$

and $\mathcal{B}_2 = \mathrm{id} - \mathcal{A}_2$. Moreover, note that $(\mathcal{A}_2, \mathcal{B}_2)$ is an integrable almost product structure on M_2. Since $\ker \mathcal{A}_2 = \ker \omega_1 \cap \ker \eta_1 \cap TM_2$ and $\mathcal{A}_2(\mathcal{R}_2) = \mathcal{R}_2$ we fix the gauge by taking the vector field

$$\xi_2 = \mathcal{R}_2 + \mathcal{A}_2\left((\bar{\mathcal{P}}X_H)_{|M_2}\right) = \frac{\partial}{\partial t} + \frac{(p_1+1)}{2}\frac{\partial}{\partial q^1} + \frac{(p_1-1)}{2}\frac{\partial}{\partial q^2} - q^2\left(\frac{\partial}{\partial p_1} + \frac{\partial}{\partial p_2}\right).$$

Now, we apply the constraint algorithm to the precosymplectic system $(\mathbb{R} \times T\mathbb{R}^3, \mathrm{d}t, \omega_L, E_L)$. The final constraint submanifold is $P_2 = P_f$, where

$$P_2 = \{(t, q^A, \dot{q}^A) \in \mathbb{R} \times T\mathbb{R}^3 \;/\; t - q^1 + q^2 = 0\}.$$

The vector field

$$\xi = \frac{\partial}{\partial t} + \frac{(\dot{q}^1 + \dot{q}^2 + 1)}{2}\frac{\partial}{\partial q^1} + \frac{(\dot{q}^1 + \dot{q}^2 - 1)}{2}\frac{\partial}{\partial q^2} - q^2\frac{\partial}{\partial \dot{q}^1}$$

is a solution of the equations of motion on P_2 and is projectable by $Leg_{|P_2}$ onto the vector field ξ_2. Following remark 6.1, we can construct a submanifold S of P_2 such that a unique solution ξ_S of the equations of motion satisfying the NSODE condition on S exists (see [5, 11]). In this case, we obtain

$$S = \{(t, q^A, \dot{q}^A) \in \mathbb{R} \times T\mathbb{R}^3 \mid t - q^1 + q^2 = 0, 1 - \dot{q}^1 + \dot{q}^2 = 0, \dot{q}^3 = 0\}$$

and

$$\xi_S = \left(\frac{\partial}{\partial t} + \dot{q}^1\frac{\partial}{\partial q^1} + (\dot{q}^1 - 1)\frac{\partial}{\partial q^2} - \frac{1}{2}q^2\left(\frac{\partial}{\partial \dot{q}^1} + \frac{\partial}{\partial \dot{q}^2}\right)\right)_{|S}.$$

Via the diffeomorphism $\alpha = (Leg_{|S})^{-1} : M_2 \longrightarrow S$ we can transport the almost product structure $(\mathcal{A}_2, \mathcal{B}_2)$ and to obtain the almost product structure $(\mathcal{A}_S, \mathcal{B}_S)$ on S given by

$$\mathcal{A}_S(\mathcal{R}_S) = \mathcal{R}_S \qquad \mathcal{A}_S\left(\frac{\partial}{\partial q^1} + \frac{\partial}{\partial q^2}\right) = \frac{\partial}{\partial q^1} + \frac{\partial}{\partial q^2}$$

$$\mathcal{A}_S\left(\frac{\partial}{\partial \dot{q}^1} + \frac{\partial}{\partial \dot{q}^2}\right) = \frac{\partial}{\partial \dot{q}^1} + \frac{\partial}{\partial \dot{q}^2} \qquad \mathcal{A}_S\left(\frac{\partial}{\partial q^3}\right) = 0$$

where

$$\mathcal{R}_S = \frac{\partial}{\partial t} + \frac{1}{2}\left(\frac{\partial}{\partial q^1} - \frac{\partial}{\partial q^2}\right).$$

7. Autonomous Lagrangian systems

Suppose that $L' : TQ \longrightarrow \mathbb{R}$ is an almost regular time-independent Lagrangian function. Let $\omega_{L'} = -\mathrm{d}(J^*(\mathrm{d}L'))$ be the Poincaré–Cartan 2-form which is supposed to be presymplectic. Let $Leg' : TQ \longrightarrow T^*Q$ be the Legendre transformation, M_1' the primary constraint submanifold on the Hamiltonian side, h_1' the Hamiltonian function on M_1', and $\omega_1' = (i')^*(\omega_Q)$, where $i' : M_1' \longrightarrow T^*Q$ is the canonical embedding.

We can consider L' as a Lagrangian function $L : \mathbb{R} \times TQ \longrightarrow \mathbb{R}$ which does not depend on the time, that is, $L(t, q^A, \dot{q}^A) = L'(q^A, \dot{q}^A)$. Therefore, L is almost regular and $(\mathrm{d}t, \omega_L)$ is a precosymplectic structure on $\mathbb{R} \times TQ$.

We can easily deduce that, if M_1 is the primary constraint submanifold, then $M_1 = \mathbb{R} \times M_1'$. Also, we obtain that the i-ary constraint submanifolds M_i and M_i' of L and L', respectively, are related by $M_i = \mathbb{R} \times M_i'$. Thus, $M_f = \mathbb{R} \times M_f'$.

The almost product structure $(\mathcal{P}', \mathcal{Q}')$ on T^*Q constructed in [14] is related with our almost product structure $(\mathcal{P}, \mathcal{Q})$ on $\mathbb{R} \times T^*Q$ by

$$\mathcal{Q} = \mathcal{Q}' \qquad \mathcal{P} = \mathcal{P}' + \mathrm{d}t \otimes \frac{\partial}{\partial t}\,.$$

Moreover, the corresponding Diract brackets are related by the following formula $\{F, G\}_{D'} = \{F, G\}_{\mathrm{D}}$, for all $F, G \in C^\infty(T^*Q)$.

8. Affine Lagrangian functions

Let $L : \mathbb{R} \times TQ \longrightarrow \mathbb{R}$ be a time-dependent Lagrangian which is affine on the velocities, i.e.

$$L(t, q^A, \dot{q}^A) = \mu_A(t, q^A)\dot{q}^A + f(t, q^A)\,.$$

L may be globally defined as follows:

$$L = \hat{\alpha} + (\tilde{\tau}_Q)^* f \tag{8.1}$$

where $\alpha = \sum_A \mu_A(t, q^B)\mathrm{d}q^A$ is a 1-form on $\mathbb{R} \times Q$ which verifies $\alpha(\partial/\partial t) = 0$, f is a function on $\mathbb{R} \times Q$, and $\hat{\alpha} : \mathbb{R} \times TQ \longrightarrow \mathbb{R}$ is the evaluation map defined by $\hat{\alpha}(t, X_q) = \alpha_{(t,q)}(X_q)$, for all $(t, X_q) \in \mathbb{R} \times T_qQ$ (see [5, 11, 12, 13]). From equation (8.1), we obtain

$$E_L = -(\tilde{\tau}_Q)^* f \qquad \omega_L = -(\tilde{\tau}_Q)^*(\mathrm{d}\alpha)\,.$$

Suppose that the pair $(\mathrm{d}t, \mathrm{d}\alpha)$ is a cosymplectic structure on $\mathbb{R} \times Q$ with Reeb vector field $\mathcal{R}$. Let E_f be the evolution vector field with respect to the cosymplectic structure $(\mathrm{d}t, \mathrm{d}\alpha)$. If $\mathcal{R}_L$ and ξ are vector fields on $\mathbb{R} \times TQ$ which are $\tilde{\tau}_Q$-projectable onto $\mathcal{R}$ and E_f, respectively, we deduce that

$$i_{\mathcal{R}_L}\omega_L = 0 \qquad i_{\mathcal{R}_L}\mathrm{d}t = 1 \qquad i_\xi\omega_L = \mathrm{d}E_L - \mathcal{R}_L(E_L)\mathrm{d}t \qquad i_\xi \mathrm{d}t = 1\,.$$

Thus, $(\mathbb{R} \times TQ, \mathrm{d}t, \omega_L, E_L, \mathcal{R}_L)$ is a precosymplectic system which admits a global dynamics. We also have that $\ker\omega_L \cap \ker \mathrm{d}t = V(\mathbb{R} \times TQ)$.

The Legendre transformation $Leg : \mathbb{R}\times TQ \longrightarrow \mathbb{R}\times T^*Q$ is given by $Leg(t, q^A, \dot{q}^A) = (t, q^A, \mu_A)$.

The 1-form α may be viewed as a time-dependent 1-form on Q, $\alpha : \mathbb{R} \times Q \longrightarrow T^*Q$ and, then, the mapping $\Psi : \mathbb{R} \times Q \longrightarrow \mathbb{R} \times T^*Q$, defined by $\psi(t, q^A) = (t, \alpha(t, q^A))$ is a diffeomorphism from $\mathbb{R} \times Q$ onto the primary constraint submanifold M_1. Thus, L is almost regular. In fact, the pair (η_1, ω_1) is a cosymplectic structure on M_1 and the map $\Psi : \mathbb{R} \times Q \longrightarrow M_1$ is a cosymplectomorphism between the cosymplectic manifolds $(\mathbb{R} \times Q, \mathrm{d}t, -\mathrm{d}\alpha)$ and (M_1, η_1, ω_1), i.e. $\Psi^*\eta_1 = \mathrm{d}t$ and $\Psi^*\omega_1 = -\mathrm{d}\alpha$ (see [5, 11]).

All the primary constraints $\Phi^A = p_A - \mu_A$, $1 \leqslant A \leqslant m$, are second class, since

$$\{\Phi^A, \Phi^B\} = \frac{\partial\mu_B}{\partial q^A} - \frac{\partial\mu_A}{\partial q^B}$$

and the matrix $\tilde{\mathcal{C}}^{AB} = (\{\Phi^A, \Phi^B\})$ is regular because $(\mathrm{d}t, \mathrm{d}\alpha)$ is cosymplectic. Then, the matrix $\mathcal{C}$ whose entries are

$$\mathcal{C}^{AB} = \frac{\partial\mu_B}{\partial q^A} - \frac{\partial\mu_A}{\partial q^B} + \frac{\partial\mu_A}{\partial t}\frac{\partial\mu_B}{\partial t}$$

is also regular. The projector $\mathcal{Q}$ is given explicitly by

$$\mathcal{Q} = \sum_{A,B} \mathcal{C}_{AB}\left(-\frac{\partial\mu_A}{\partial t}\frac{\partial}{\partial t} + \frac{\partial}{\partial q^A} + \sum_C \frac{\partial\mu_A}{\partial q^C}\frac{\partial}{\partial p_C}\right) \otimes \left(\mathrm{d}p_B - \frac{\partial\mu_B}{\partial q^D}\mathrm{d}q^D - \frac{\partial\mu_B}{\partial t}\mathrm{d}t\right)$$

and $\mathcal{P} = \mathrm{id} - \mathcal{Q}$. We also have that

$$\mathcal{R} = \frac{\partial}{\partial t} + \sum_{A,B} \mathcal{C}_{AB} \frac{\partial \mu_B}{\partial t} \frac{\partial}{\partial q^A} + \sum_{A,B,C} \mathcal{C}_{AB} \frac{\partial \mu_B}{\partial t} \frac{\partial \mu_A}{\partial q^C} \frac{\partial}{\partial p_C}$$

$$\eta = \mathrm{d}t - \sum_{A,B,D} \mathcal{C}_{AB} \frac{\partial \mu_A}{\partial t} \frac{\partial \mu_B}{\partial q^D} dq^D + \sum_{A,B} \mathcal{C}_{AB} \frac{\partial \mu_A}{\partial t} \mathrm{d}p_B \, .$$

The Dirac bracket $\{\ ,\ \}_{\mathrm{D}}$ on $\mathbb{R} \times T^*Q$ is given by

$$\begin{aligned}\{F, G\}_D = {} & \sum_{A=1}^{m} \left(\frac{\partial F}{\partial q^A} \frac{\partial G}{\partial p_A} - \frac{\partial G}{\partial q^A} \frac{\partial F}{\partial p_A} \right) \\ & + \sum_{A,B,C,D=1}^{m} \mathcal{C}_{AB} \left(\frac{\partial G}{\partial q^B} + \frac{\partial \mu_B}{\partial q^C} \frac{\partial G}{\partial p_C} \right) \left(\frac{\partial F}{\partial q^A} + \frac{\partial \mu_A}{\partial q^D} \frac{\partial F}{\partial p_D} \right) \\ & - \sum_{A,B,C,D,A',B'=1}^{m} \mathcal{C}_{AB} \mathcal{C}_{A'B'} \left(\frac{\partial F}{\partial q^B} + \frac{\partial \mu_B}{\partial q^C} \frac{\partial F}{\partial p_C} \right) \left(\frac{\partial G}{\partial q^{B'}} + \frac{\partial \mu_{B'}}{\partial q^D} \frac{\partial G}{\partial p_D} \right) \frac{\partial \mu_A}{\partial t} \frac{\partial \mu_{A'}}{\partial t} \, .\end{aligned}$$

If H is an extension of h_1 to $\mathbb{R} \times T^*Q$, the restriction of the $\mathcal{P}$-evolution vector field $E_{(H,\mathcal{P})}$ is the solution of the dynamics (see theorem 5.2). Moreover, $(E_{(H,\mathcal{P})})_{|M_1}$ is just the vector field $T\Psi(E_f)$ (see [5]). Finally, in this case, the submanifold S of $\mathbb{R} \times TQ$ is $E_f(\mathbb{R} \times Q)$, and the vector field ξ_S is the restriction to S of the complete lift of E_f to $T(\mathbb{R} \times Q)$ (see [5]).

Acknowledgments

This research was partially supported by DGICYT, Project PB94-0106, Consejería de Educación del Gobierno de Canarias-Spain and UNED-Spain.

Appendix

We will prove that the determinant of the matrix $\mathcal{C}$ whose entries are $\mathcal{C}^{ab} = \{\Phi^a, \Phi^b\} + (\partial \Phi^a / \partial t) \partial \Phi^b / \partial t$ coincides with the determinant of the matrix $\tilde{\mathcal{C}}^{ab} = \{\Phi^a, \Phi^b\}$. The result follows from the following propositon.

Proposition A.1. Let $A = (a_{ij})$ be a skew-symmetric matrix where $1 \leqslant i, j \leqslant n$, and n is an even number, and let $B = (b_{ij})$ be a symmetric matrix defined by $b_{ij} = b_i b_j$, $b_i, b_j \in \mathbb{R}$, $1 \leqslant i, j \leqslant n$. We deduce that

$$|A + B| = |A| \, .$$

Proof. We have that

$$\begin{aligned}|A + B| = {} & \sum_{\sigma \in S_n} (-1)^\sigma (a_{1\sigma(1)} + b_1 b_{\sigma(1)})(a_{2\sigma(2)} + b_2 b_{\sigma(2)}) \cdots (a_{n\sigma(n)} + b_n b_{\sigma(n)}) \\ = {} & |A| + \sum_{i=1}^{n} \sum_{\sigma \in S_n} (-1)^\sigma b_i b_{\sigma(i)} a_{1\sigma(1)} \cdots \widehat{a_{i\sigma(i)}} \cdots a_{n\sigma(n)} \\ & + \sum_{i_1, i_2 = 1, \ldots, n, i_1 < i_2} \sum_{\sigma \in S_n} (-1)^\sigma b_{i_1} b_{\sigma(i_1)} b_{i_2} b_{\sigma(i_2)} a_{1\sigma(1)} \cdots \widehat{a_{i_1\sigma(i_1)}} \cdots \widehat{a_{i_2\sigma(i_2)}} \cdots a_{n\sigma(n)}\end{aligned}$$

$$+\cdots+\sum_{i_1,\ldots,i_{n-1}=1,\ldots,n,i_1<\cdots<i_{n-1}}\sum_{\sigma\in S_n}(-1)^\sigma b_{i_1}b_{\sigma(i_1)}\cdots b_{i_{n-1}}b_{\sigma(i_{n-1})}a_{i_n\sigma(i_n)}$$

$$+\sum_{\sigma\in S_n}(-1)^\sigma b_1 b_{\sigma(1)}\cdots b_n b_{\sigma(n)}\,.$$

All the terms, with exception of the first and second ones, are trivially equal to zero. Therefore

$$|A+B| = |A| + \sum_{i=1}^{n} b_i(-1)^{1+i}\left(\sum_{\sigma\in S_n}(-1)^\sigma(-b_{\sigma(1)})a_{1\sigma(2)}\cdots a_{i-1\,\sigma(i)}a_{i+1\,\sigma(i+1)}\cdots a_{n\sigma(n)}\right)$$

$$= |A| + |C|$$

where C is the skew-symmetric matrix defined by $c_{11}=0$, $c_{1j}=b_{j-1}$ $(j>2)$, $c_{i1}=-b_{i-1}$ $(i>2)$ and $c_{ij}=a_{i-1,j-1}$ $(i,j>2)$. Since C is a skew-symmetric matrix of odd order, then $|C|=0$. Thus, $|A+B|=|A|$. □

References

[1] Albert C 1989 Le théorème de réduction de Marsden–Weinstein en géométrie cosymplectique et de contact *J. Geom. Phys.* **6** 627–49

[2] Bhaskara K H and Viswanath K 1988 *Poisson Algebras and Poisson Manifolds* (London: Longman Scientific and Technical)

[3] Cantrijn F, de León M and Lacomba E A 1992 Gradient vector fields on cosymplectic manifolds *J. Phys. A: Math. Gen.* **25** 175–88

[4] Cariñena J F 1990 Theory of singular Lagrangians *Fort. Phys.* **38** 641–79

[5] Chinea D, de León M and Marrero J C 1994 The constraint algorithm for time-dependent Lagrangians *J. Math. Phys.* **35** 3410–47

[6] Dirac P A M 1964 *Lecture on Quantum Mechanics* (New York: Belfer Graduate School of Science, Yeshiva University)

[7] Dubrovin B A, Giordano M, Marmo G and Simoni A 1993 Poissons brackets on presymplectic manifolds *Int. J. Mod. Phys.* A **8** 3747–71

[8] Gotay M J 1979 Presymplectic manifolds, geometric constraint theory and the Dirac–Bergmann theory of constraints *Dissertation* Center for Theoretical Physics, University of Maryland

[9] Gotay M J and Nester J M 1979 Presymplectic Lagrangian systems I: the constraint algorithm and the equivalence theorem *Ann. Inst. H Poincaré* A **30** 129–42

[10] Gotay M J and Nester J M 1980 Presymplectic Lagrangian systems II: the second-order differential equation problem *Ann. Inst. H Poincaré* A **32** 1–13

[11] de León M, Marín-Solano J and Marrero J C 1996 The constraint algorithm in the jet formalism *Diff. Geom. Appl.* to appear

[12] de León M and Marrero J C 1993 Constrained time-dependent Lagrangian systems and Lagrangian submanifolds *J. Math. Phys.* **34** 622–44

[13] de León M and Marrero J C 1993 Time-dependent linear Lagrangians: the inverse problem, symmetries and constants of motion *Proc. Int. Symp. on Hamiltonian Systems and Celestial Mechanics, Guanajuato, Mexico, September 30–October 4, 1991, (Advanced Series in Nonlinear Dynamics 4)* (Singapore: World Scientific) pp 55-83

[14] de León M, Martín de Diego D and Pitanga P 1995 A new look at degenerate Lagrangian dynamics from the viewpoint of almost product structures *J. Phys. A: Math. Gen.* **28** 4951–71

[15] de León M and Rodrigues P R 1989 *Methods of Differential Geometry in Analytical Mechanics (Mathematics Series 152)* (Amsterdam: North-Holland)

[16] Marmo G, Mukunda N and Samuel J 1983 Dynamics and symmetry for constrained systems: a geometrical analysis *Riv. Nuovo Cimento* **6** 1–62

[17] Sato I 1976 On a structure similar to the almost contact structure *Tensor, N. S.* **30** 219–24

[18] Sudarshan E C G and Mukunda N 1974 *Classical Dynamics: A Modern Perspective* (New York: Wiley)

[19] Sundermeyer K *Constrained Dynamics (Lecture Notes in Physics 169)* (Berlin: Springer)

[20] Wipf A 1994 *Hamilton's Formalism for Systems with Constraints (Lecture Notes in Physics 434)* (Berlin: Springer)